Fuzzy Sets - Introduction

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If you only have a hammer, everything looks like a nail.

Summary

- Classical Sets
- Classical Sets and Membership Functions
- Classical Sets Operations
- Classical Set Properties
- Fuzzy Sets
- Membership Functions and Fuzzy Sets
- Fuzzy Sets Terminology

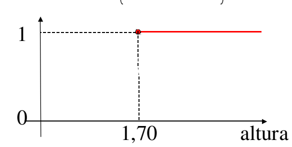
Crips Sets

- Universe of Discourse
  - It is the space where all set elements are defined
  - For instance:
    - **Human heights: 0 <= h <= 2.5m**
    - **room temperatures: -70°<=temp<=70°**

Classical Sets

- Membership Function
  - Defines whether a given element belongs to a set

$$\chi_A(X) = \begin{Bmatrix} 1 & x \geq 1.70 \\ 0 & x < 1.70 \end{Bmatrix}$$



Classical Sets/Problems

- Classical sets have problems when applied to a wide class of real problems.
- The difficult problem of defining the frontier between two sets (e.g. high/ not high) is called Sorites Paradox, due to the dialectic Eubulides from Miletus, opponent of Aristotle.
- The paradox is stated as follows:

"Is one grain of sand a heap? Are two grains of sand a heap? ..."

Truth Tables

A	B	$A \cup B$	$A \cap B$	$\overline{A}$
0	0	0	0	1
0	1	1	0	1
1	0	1	0	0
1	1	1	1	0

$1 \rightarrow true$

$0 \rightarrow false$

Which mathematical functions could be used to represent the union, intersection and complement?

Classical Sets Properties

Involution	$\overline{\overline{A}}=A$
Comutatividade	$A\cup B=B\cup A$
	$A\cap B=B\cap A$
Associatividade	$A\cup(B\cup C)=(A\cup B)\cup C$
	$A\cap(B\cap C)=(A\cap B)\cap C$
Distributividade	$A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$
	$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

Classical Sets Properties

Idempotência	$A\cup A=A$
	$A\cap A=A$
Identidade	$A\cup\emptyset=A$
	$A\cap\emptyset=\emptyset$
	$A\cup X=X$
	$A\cap X=A$
De Morgan	$\overline{A\cap B}=\overline{A}\cup\overline{B}$
	$\overline{A\cup B}=\overline{A}\cap\overline{B}$

The Laws of Aristotle

- "Everything must be or not be, either in the present or in the future."
- The Union of a set to its complement yields the Universe.

$$A\cup\overline{A}=X$$

- This is called the *law of the excluded middle*.

The Laws of Aristotle

- The intersection of a set and its complement is empty.

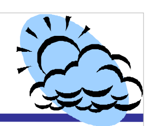
$$A\cap\overline{A}=\emptyset$$

- This is called the *law of non-contradiction*.

And now for something completely different!

Thanks to

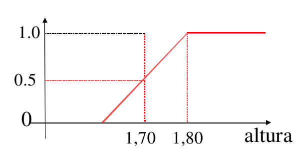
Fuzzy Sets



- The membership function of elements in a fuzzy set A is characterised by

$$\mu(.):X\rightarrow[0,1]$$

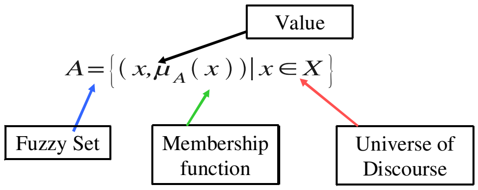
that maps every element from the set X into a real number in the interval [0,1].



Fuzzy Sets - definition

- A fuzzy set can be represented by an ordered set of pairs:

$$A = \{ (x, \mu_A(x)) \mid x \in X \}$$



Membership degree

- **Membership degree**: an element belongs to a given set by a degree of certainty.
- Some elements are more representative of the set main idea than others.
  - *Excellent students={(Pedro,0.8), (Ana,0.9), (Paulo,0.9), (Marta,1.0)}*
  - *Very high = {(Oscar,0.95), (Michael Jordan, 0.95), (Junior Baiano,0.8)}*

Membership

- One element may belong partially to more than one set
  - *children={Pedro, Ana, Paulo, Marta}*
  - *adolescents = {Pedro, Mateus, Joaquim}*
  - *children(Pedro)=0.2*
  - *adolescents(Pedro)=0.8*

It isn't probability!

- Belonging to the set of high people with a membership degree of 0.25 indicates separation from the ideal definition of high people by 0.75.
- A degree value of 0.25 does not mean that a person this high may be found with probability 0.25 in the universe of people.

It isn't probability!

- A bottle full of an unknown liquid has 95% probability of being pure poison.
- Another full bottle of a very similar liquid belongs to the set of pure water to a degree of 0.95, which means 5% poison.
- This concentration of poison does not kill, however you will be very sick, may be a doctor will be required.
- ***If you know which bottle to choose, you understood the differences.***

Fuzzy Sets Representation 1

- **Ordered Pairs**: A fuzzy set may represented by a set ordered pairs
- The first element is the element itself and the second its membership degree in the set.
- João = 1.65 m; Ana = 1.70 m; Oscar= 1.80 m

High = {(João, 0.25), (Ana,0.5), (Oscar,1.0)}

Fuzzy Sets Representation 2

- Indicated as the union of all elements

$$A = \sum_{x_i \in A} \mu_A(x_i)/x_i$$

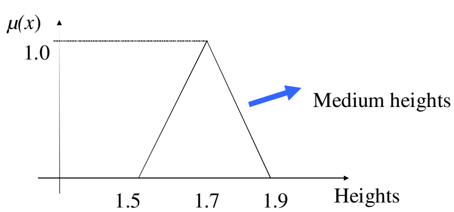
High = 0.25/João + 0.5/Ana + 1.0/Oscar

Fuzzy Sets Representation 3

- **Membership function: a fuzzy set may be represented by a function that maps its elements into real numbers in the interval [0,1].**

$$\mu_{media}(altura) = \begin{cases} 0 & altura \leq 1.5 \\ 5 \times altura - 7.5 & 1.5 \leq altura \leq 1.7 \\ -5 \times altura - 9.5 & 1.7 < altura < 1.9 \\ 0 & altura \geq 1.9 \end{cases}$$

Example of Membership Functions

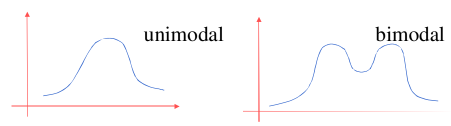


Unimodal Functions

- A function is **unimodal** if

$$\forall x_1, x_2 \in X, \forall \lambda \in [0,1]: \mu(\lambda x_i + (1-\lambda) x_2) \geq \min[\mu(x_1), \mu(x_2)]$$

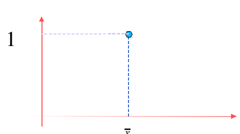
- An unimodal function implies that whenever *μ(x)>μ(y)* for a given set *A* then *x* is closer to the ideal definition of *A* than *y*.



Singletons

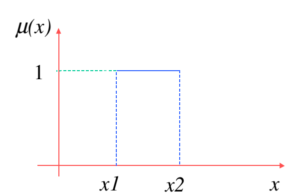
- A Function is a **singleton** whenever

$$\mu_{\bar{x}}(x) = \begin{cases} 1 & x = \bar{x} \\ 0 & x \neq \bar{x} \end{cases}$$



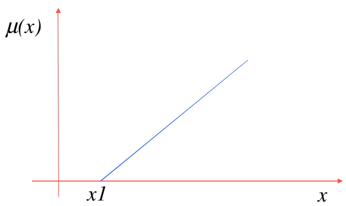
Classical Set Function

- Classical set functions are used to define Classical sets.



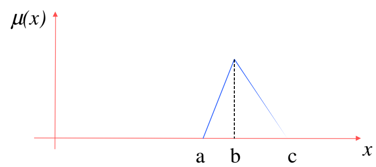
Linear Functions

- Very simple fuzzy set.



Triangular Functions

- Easy to implement, allowing representation of very complex fuzzy sets.
- May be generated from 3 real values.



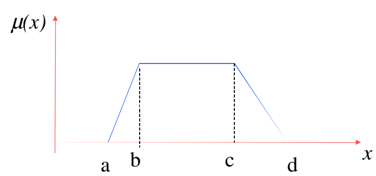
Triangular Functions

- May be generated from 3 real values.

$$tri(x;a,b,c)=\begin{Bmatrix} 0 & x<a \\ (x-a)/(b-a) & a\le x\le b \\ (c-x)/(c-b) & b\le x\le c \\ 0 & x>c \end{Bmatrix}$$

Trapezoidal Functions

- Easy to implement, allowing representation of very complex fuzzy sets.
- May be generated from 4 real values.



Trapezoidal Functions

- May be generated from 4 real values.

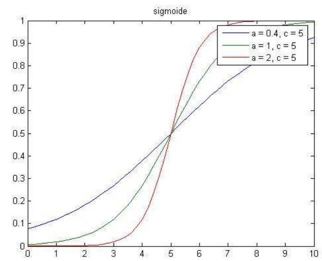
$$trap(x:a,b,c,d)=\begin{Bmatrix} 0 & x<a \\ \frac{x-a}{b-a} & a\le x<b \\ 1 & b\le x<c \\ \frac{d-x}{d-c} & c\le x<d \\ 0 & x\ge d \end{Bmatrix}$$

Sigmoid Function

- It is described by two parameters: the inflexion point (c), that is the point where the function is equal to 0.5 and a parameter that defines the shape.

$$Sig(x;a,c)=\left\{\frac{1}{1+e^{-a(x-c)}}\right\}$$

Sigmoid Function

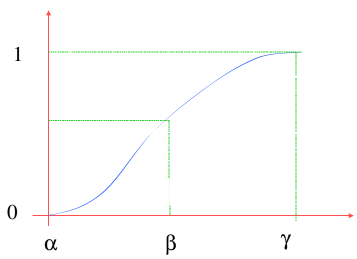


S Function

- It is described by three parameters: its 0 degree value (α), its 1 degree value (γ) and the inflexion point (β), that is the point where the function is equal to 0.5 (β=(α +γ)/2).

$$S(x,\alpha,\beta,\gamma)=\begin{cases} 0 & x\le\alpha \\ 2\times\left(\frac{x-\alpha}{\gamma-\alpha}\right)^2 & \alpha<x\le\beta \\ 1-2\times\left(\frac{x-\alpha}{\gamma-\alpha}\right)^2 & \beta<x\le\gamma \\ 1 & x>\gamma \end{cases}$$

S Function

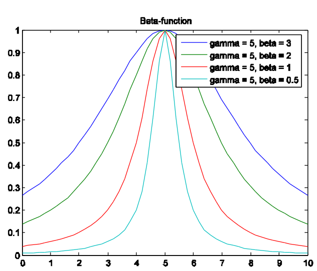


Beta Function

- It is defined by two parameters, the value where the function is equal to one (γ) and the inflexion (β).

$$B(x,\gamma,\beta)=\frac{1}{1+\left(\frac{x-\gamma}{\beta}\right)^2}$$

Beta Function



Support of fuzzy sets

- The **support of a fuzzy set** *A,* defined in the universe of discourse *X,* is the classical set defined as

$$S_A=\{x\in X\,|\,\mu_A(x)>0\}$$

Empty Fuzzy Set

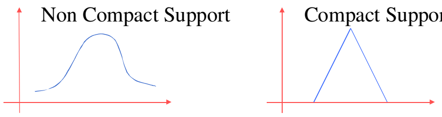
- A fuzzy set ($A=\varnothing$) is **empty** if its membership function is zero everywhere in its universe of discourse.

$$A=\varnothing \quad if \;\; \mu_A(x)=0, \forall x \in X$$

- An **empty fuzzy set** has an **empty support**

Compact Support

- The support is compact when the set is smaller than the size of the universe of discourse.
- If the support were not compact then several rules would be activated at every input causing an increase in the system load.



Alpha-Cuts Sets

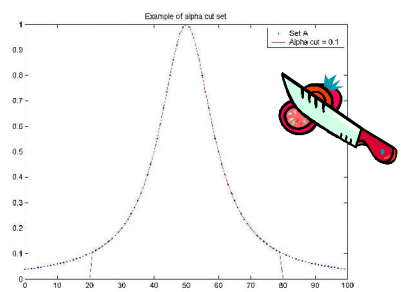
- The **classical set A_α, called alpha-cut set**, is the set of elements whose degree of membership in *A* is no less than α. It is defined as:

$$A_\alpha=\{x \in X \mid \mu_A(x) \geq \alpha\}$$

- The **classical set A'_α is called strong alpha-cut** set. It is defined as:

$$A'_\alpha=\{x \in X \mid \mu_A(x) > \alpha\}$$

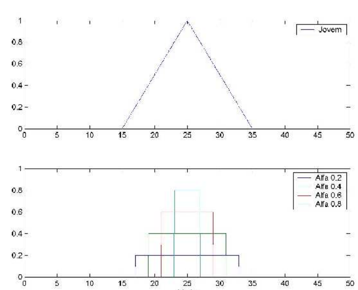
Alpha-cut Set



Example of Alpha-cuts

- Universe of Discourse (Ages): {5,10,20,…,80}
- *Young* = {(15,0), (17, 0.2), (19, 0.4), (21,0.6), (23, 0.8), (25,1.0), (27, 0.8), (29, 0.6), (31,0.4), (33, 0.2), (35,0.0)}
- $Young_{02}$ = {17, 19, 21, 23, 25, 27, 29, 31, 33}
- $Young'_{02}$ = {19, 21, 23, 25, 27, 29, 31}

Alpha-cuts



Level Set

- The set of all levels alpha that represent distinct alpha-cuts of a given fuzzy set *A* is called a level set of *A*.

$$\Lambda_A = \{\alpha \mid \mu_A(x) = \alpha \ for\ some\ x \in X\}$$

$$Ex.$$

$$\Lambda_{young} = \{0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0\}$$

Resolution Principle

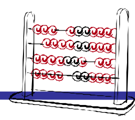
- The membership function of a fuzzy set *A* can be expressed in terms of its alpha cuts as (+ is union):

$$\Lambda = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$$

$$A = \alpha_0 \times A_{\alpha_0} + \alpha_0 \times A_{\alpha_1} + \cdots + \alpha_n \times A_{\alpha_n}$$

$$\mu_{\alpha_0 \times A_{\alpha_0}} = \begin{cases} \alpha_i & if\ \mu_A(x) \geq \alpha_i \\ 0 & otherwise \end{cases}$$

Cardinality



- The cardinality |*A*| of a fuzzy set *A* is defined as

$$|A| = \sum_{x \in X} \mu(x)$$

- The relative cardinality of *A* is defined as

$$\|A\| = \frac{|A|}{|X|}$$

Cardinality - Example

Consider the set

A = {(6.5, 0.25), (7,0.5), (7.5,0.75), (8,1), (8.5,0.75), (9,0.5), (9.5,0.75)}

defined in the universe of grades from 0 to 10 in 0.5 steps. The cardinality of *A* is

$$|A| = 0.25 + 0.5 + 0.75 + 1 + 0.75 + 0.5 + 0.25 = 4.0$$

So the relative cardinality of *A* is

$$\|A\| = \frac{4.0}{21} = 0.2$$

Cardinality - cont

- When the fuzzy set is continuous the cardinality is defined as

$$|A| = \int_x \mu_A(x)\,dx$$

Fuzzy Cardinality

- When a fuzzy set A has a finite support, its fuzzy cardinality |Ã| is a fuzzy set (fuzzy number) defined on N whose membership function is defined by

$$\mu_{|\tilde{A}|}(\alpha) = |A_\alpha| \forall\ \alpha \in \Lambda_A$$

Fuzzy Cardinality Example

- *Young* = {(15,0), (17, 0.2), (19, 0.4), (21,0.6), (23, 0.8), (25,1.0), (27, 0.8), (29, 0.6), (31,0.4), (33, 0.2), (35,0.0)}
- A_{yarg} = {0.2, 0.4, 0.6, 0.8, 1.0}
- $Young_{02}$ = {17, 19, 21, 23, 25, 27, 29, 31, 33}
- $|Young_{02}|$ = 9
- $Young_{04}$ = {19, 21, 23, 25, 27, 29, 31}
- $|Young_{04}|$ = 7

Fuzzy Cardinality Example

- *Young* = {(15,0), (17, 0.2), (19, 0.4), (21,0.6), (23, 0.8), (25,1.0), (27, 0.8), (29, 0.6), (31,0.4), (33, 0.2), (35,0.0)}

$$|\tilde{Young}| = 0.2/9 + 0.4/7 + 0.6/5 + 0.8/3 + 1.0/1$$

Height of a fuzzy set

- The **height** of a fuzzy set is the **highest membership** value of its membership function

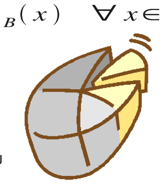
$$H_A = \max_{x \in X}\{\mu_A(x)\}$$

- A fuzzy set is defined as normal when H_A=1 and subnormal when it is not.

Fuzzy Subsets

- If the membership grade of each element of the fuzzy set A is less than or equal its membership grade in fuzzy set B, then A is called a subset of B.

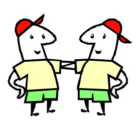
$$A \subseteq B \quad if \quad \mu_A(x) \le \mu_B(x) \quad \forall\, x \in X$$



Equal Fuzzy Sets

- If the membership grade of each element of the fuzzy set A is equal its membership grade in fuzzy set B, then A is equal to B.

$$A = B \quad if \quad \mu_A(x) = \mu_B(x)$$

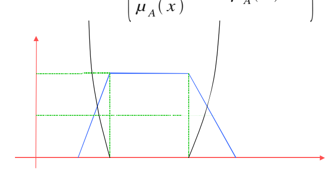


$$A = B \ if \ \ A \subseteq B \ and \ B \subseteq A$$

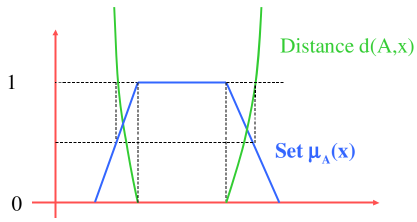
Distance

- Measures the distance that a value is from the ideal definition of the set. It is defined as

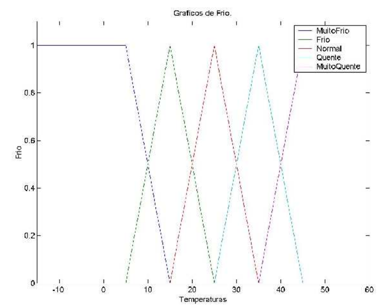
$$d(A,x) = \begin{cases} \infty & \mu_A(x) = 0 \\ \dfrac{1}{\mu_A(x)} - 1 & \mu_A(x) \neq 0 \end{cases}$$



Distance



Examples of Fuzzy Sets



Examples of Fuzzy Sets